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A Numerical Examination of Certain Null Geodesics of a High Angular Momentum Kerr Field

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Abstract

An observer situated anywhere but in the equatorial plane of a high angular momentum Kerr field cannot see the ring singularity. In the visual field of such an observer, what demarcates his own universe from that through the ring?

The projections onto a certain submanifold of the null geodesics which pass through a point on the symmetry axis of a specific Kerr field are examined numerically. All the distinct projections are obtained by varying one parameter, essentially the quadratic Killing tensor constant. Various interesting features of the geodesics emerge.

Through the ring is a region in which there exist closed time-like curves and which can be used to construct closed time-like curves through any non-singular point of the manifold. Only geodesics of negative angular momentum can enter this region.

It has been shown by Carter (1968) that only equatorial geodesics of a Kerr field (Kerr, 1963; Kerr & Schild, 1964) can reach the singularity. The problem therefore arises as to what exactly an observer (potentially) sees who is not in the equatorial plane. Consider, for example, an observer on the symmetry axis of the field in the 'positive r sheet'. He will receive some photons from his own familiar universe and others from the region through the ring but none from the ring itself. In the visual field of such an observer, what demarcates his own universe from the region on the other side of the ring? Surely there must be directions from which photons cannot arrive. In fact there are and the set of such directions does constitute the boundary between the two regions in the visual field of the observer. However, the detailed behaviour of the null geodesics which reach our observer is interesting and it is worth looking at them in a particular numerical case.

In Kerr–Newman coordinates (u, r, θ, ϕ) (Boyer & Lindquist, 1967; Carter, 1968), by considering the Jacobi action, first-order geodesic equations can be obtained directly (Carter, 1968).

$$(r^2 + a^2 \cos^2\theta)^2 \dot{\theta}^2 = Q + \cos^2\theta[a^2(E^2 - \mu^2) - \Phi^2 \operatorname{cosec}^2\theta] \tag{1}$$

$$(r^2 + a^2\cos^2\theta)^2 \dot{r}^2 = [E(r^2 + a^2) - \Phi a]^2 - \Delta(\mu^2 r^2 + K),$$
$$\Delta = r^2 - 2mr + a^2 \tag{2}$$

Here $E = p_0$, $\Phi = -p_3$, $(\cdot)$ denotes differentiation w.r.t. proper time or an affine parameter, λ, and $Q = K - (\Phi - aE)^2$ where K is a constant of the motion originally obtained from the separability of the Hamilton Jacobi equations (Carter, 1968) and which is associated with the quadratic Killing tensor of Walker & Penrose (1970). E, Φ and K can be identified with the energy of the particle as measured at infinity, with its angular momentum about the symmetry axis and the square of its total angular momentum respectively, as measured at infinity. It is apparent from (1) that a geodesic can pass through a point on the symmetry axis $\theta = 0$ only if $\Phi = 0$. So, for the null geodesics we are concerned with, (1) and (2) become:

$$(r^2 + a^2\cos^2\theta)^2 \dot{\theta}^2 = K - a^2 E^2 \sin^2\theta \tag{3}$$

$$(r^2 + a^2\cos^2\theta)^2 \dot{r}^2 = E^2(r^2 + a^2)^2 - K\Delta \tag{4}$$

We are considering a field in which $a > m$ so Δ is strictly positive. Therefore, if $E = 0$, (3) and (4) give opposite signs for $\dot{\theta}^2$ and $\dot{r}^2$ on $\theta = 0$. Hence no null geodesics with $E = 0$ can pass through a point on $\theta = 0$. Choosing a new affine parameter $\lambda' = E\lambda$, and replacing K with E^2K, (3) and (4) become:

$$(r^2 + a^2\cos^2\theta)^2 \dot{\theta}^2 = K - a^2 \sin^2\theta \tag{5}$$

$$(r^2 + a^2\cos^2\theta)^2 \dot{r}^2 = (r^2 + a^2)^2 - K\Delta \tag{6}$$

where $(\cdot)$ now denotes differentiation w.r.t. λ'. Thus eliminating λ', the projections of the null geodesics onto an (r, θ) submanifold are given by the solutions of

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{(r^2 + a^2)^2 - K\Delta}{K - a^2\sin^2\theta} \tag{7}$$

where we will trace them away from our observer at $\theta = 0$, $r = r_0$. The solution of (7) by separating variables involves incomplete elliptic integrals of the first kind.

We considered a field for which $a = 2m$. With this (7) becomes:

$$\frac{dr}{d\theta} = \pm\sqrt{\frac{R}{\Theta}} \tag{8}$$

where $R = (r^2 + 4m^2)^2 - K(r^2 - 2mr + 4m^2)$ and $\Theta = K - 4m^2 \sin^2 \theta$.

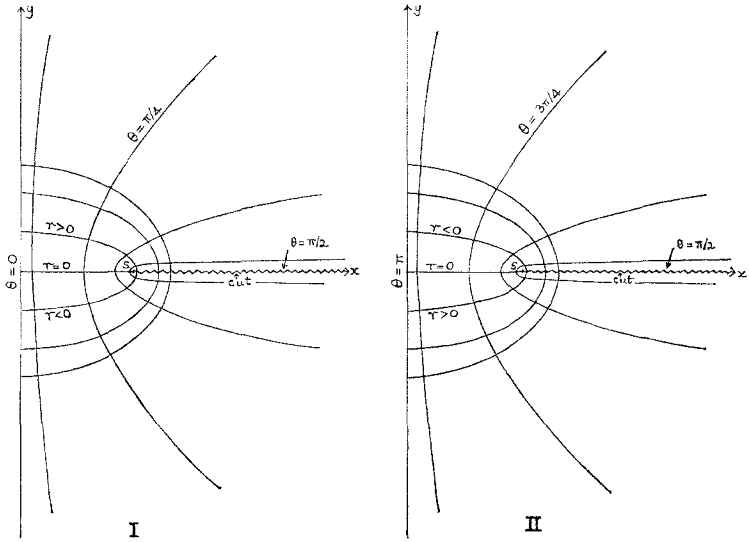

Figure 1—The (r, θ) space.

Solutions of (8) were computed numerically for a selection of values of K between 0 and $8m^2$ starting from the initial point $\theta_0 = 0$, $r_0 = 8m$. The solutions were viewed in a space of cartesian coordinates (x,y) where

$$x = (r^2 + 4m^2)^{1/2} \sin\theta, \qquad y = r\cos\theta \tag{9}$$

(x,y) were obtained from the Kerr–Schild coordinates (T, X, Y, Z) by:

$$x = \sqrt{(X^2 + Y^2)}, \qquad y = Z \tag{10}$$

the transformation between the Kerr–Newman and Kerr–Schild coordinates being (Boyer & Linquist, 1967; Carter, 1968)

$$X + iY = (r + ia)\,e^{i\phi} \sin\theta, \qquad Z = r\cos\theta, \qquad T = u - r \tag{11}$$

The (r, θ) space consists of two sheets I and II joined along Sx, where S is the projection of the ring singularity. The r = constant and θ = constant lines are parts of ellipses and hyperbolas with focus, S. S is given by $\theta = \pi/2$, $r = 0$, i.e. $x = 2m$, $y = 0$ (Fig. 1).

The computation was handled using an *ad hoc* modification of the Runge–Kutta routine to cope with the fact that we run into zeros of R and Θ. The solutions were started with the negative square root in (8) in order that they should initially approach the ring. When a zero of R is encountered the sign in (8) has to be changed and when a zero in Θ is approached (to within certain specified limits) the increments of θ must be made negative. Zeros of R and Θ can be encountered on the same curve, so a curve can

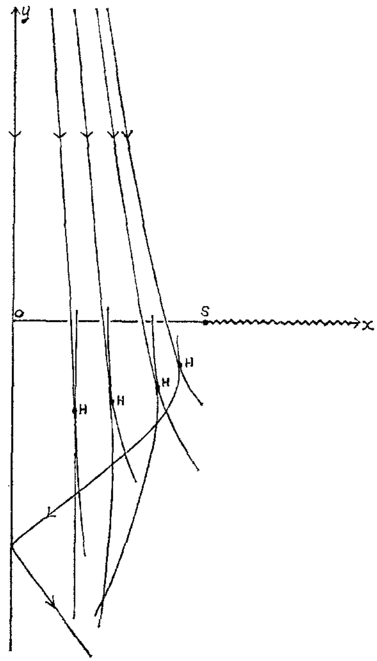

Figure 2—Projections of null geodesics with $K < K_1$.

'bounce off' both a hyperbola and an ellipse. As K is increased, no zero of R occurs until the double root $r = r_1$. This is the real root of

$$r^3 - 3mr^2 + 4m^2 r + 4m^3 = 0 \tag{12}$$

and is approximately $-0{\cdot}6344m$. The corresponding value of K is:

$$K_1 = 2r_1 \frac{(r_1^2 + 4m^2)}{r_1 - m} \doteqdot 3{\cdot}4176m^2 \tag{13}$$

Zeros of Θ are met for $K < 4m^2$ at $\theta = \sin^{-1}(K^{1/2}/2m)$. Referring to Fig. 2, $K = 0$ gives the solution $\theta = 0$. Increasing K from 0, the projections of the null geodesics are at first nearly straight lines (we must bear in mind of course that the null geodesics themselves are spiralling in the ϕ direction, this occurring most rapidly near the ring). But as the critical value K_1 is approached the solutions are bent back more and more until they start to approach the axis. The reflection at the axis is a consequence of the projection. When K is just below K_1, the solution bounces off the hyperbola $\theta = \sin^{-1}(K^{1/2}/2m)$ and is reflected off the axis a number of times. Finally, it is reflected off the axis at such an angle that it does not meet the hyperbola again (Fig. 3), the solution being by now indistinguishable from a straight

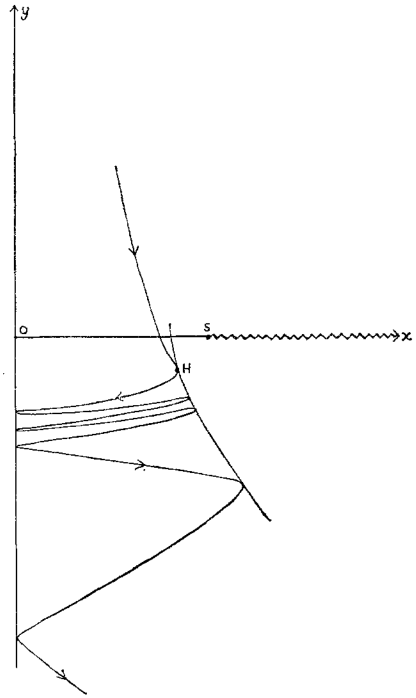

Figure 3—K approaches K_1.

line. The corresponding geodesic is spiralling into the symmetry axis and out again each time it meets $\theta = \sin^{-1}(K^{1/2}/2m)$. The 'zig-zags' of the solution close up at first after H and then open out again.

For $K = K_1$ the zig-zags close up completely, r decreases asymptotically to r_1. The null geodesic never reaches $r = r_1$ no matter how large a lapse of the affine parameter takes place. Yet from (5) we see that $\dot{\theta}^2$ must approach $(K_1 - 4m^2 \sin^2 \theta)/(r_1^2 + 4m^2 \cos^2 \theta)$ uniformly and therefore does not go to zero for all θ: $0 \leqslant \theta \leqslant \sin^{-1}(K_1^{1/2}/2m)$. Furthermore, $\sin^{-1}(K_2^{1/2}/2m)$ is not a double root of $K_1 - 4m^2 \sin^2 \theta = 0$. Hence θ goes on changing indefinitely and the null geodesic ends up spiralling in and out between $\theta = 0$ and $\theta = \sin^{-1}(K_1^{1/2}/2m)$ virtually on the surface $r = r_1$ (Fig. 4).

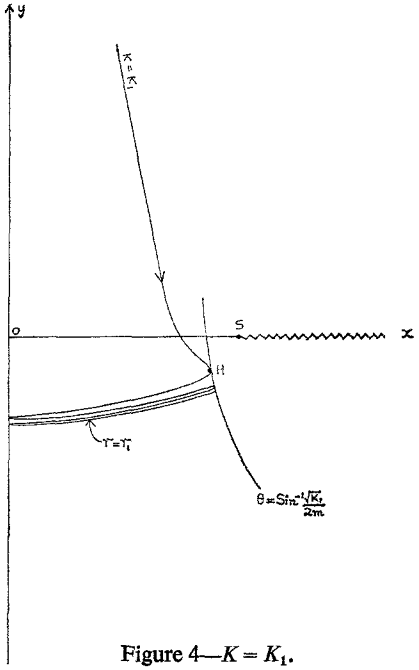

Figure 4—$K = K_1$.

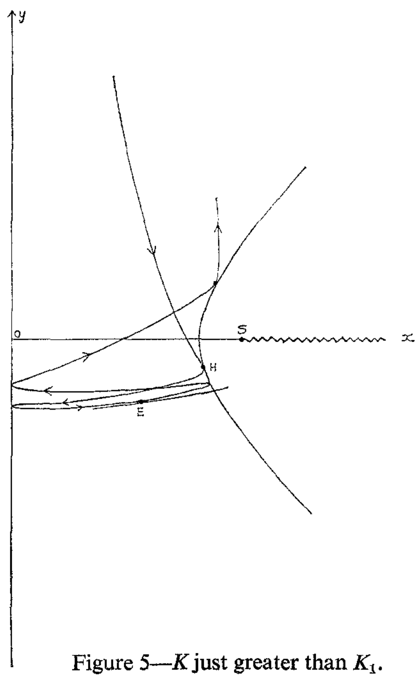

Figure 5—K just greater than K_1.

For $K < K_1$ all solutions end up on the negative sheet. For $K > K_1$ all solutions end up on the positive sheet, since R then encounters a zero. Running the solutions backwards, photons with $K < K_1$ arrive at our observer from the other side of the ring; with $K > K_1$ from his own side of the ring. From directions towards the ring in which photons with $K = K_1$ would head, none arrive. These directions would constitute a circle in the observer's visual field.

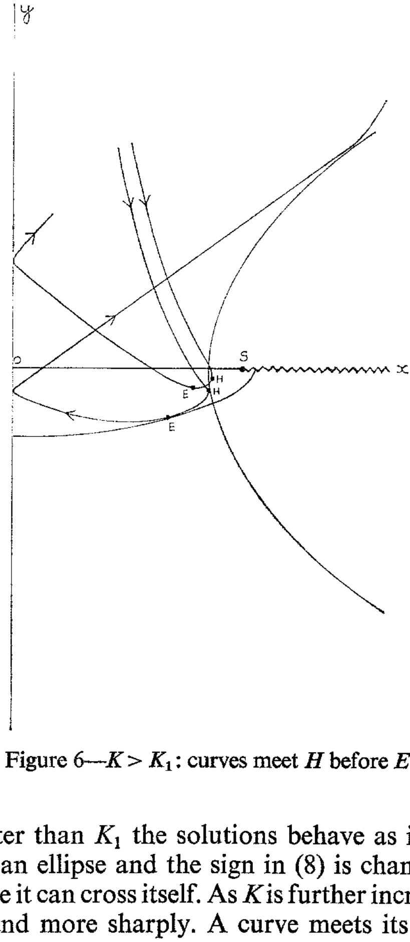

Figure 6—$K > K_1$: curves meet H before E.

For K just greater than K_1 the solutions behave as in Fig. 5. At E the curve bounces off an ellipse and the sign in (8) is changed for the rest of the solution—hence it can cross itself. As K is further increased the solutions bend back more and more sharply. A curve meets its 'minimum ellipse' sooner and sooner after meeting its 'maximum hyperbola' (Fig. 6). A second critical value of K, found by numerical search, is $K_2 \doteqdot 3{\cdot}580m^2$. The solution for $K = K_2$ meets H and E simultaneously and the curve ceases to be regular at this point and retraces itself in the opposite direction. The zeros of R and Θ at the point are not double so the null geodesic goes back through the initial point (Fig. 7). There is no possibility of it returning

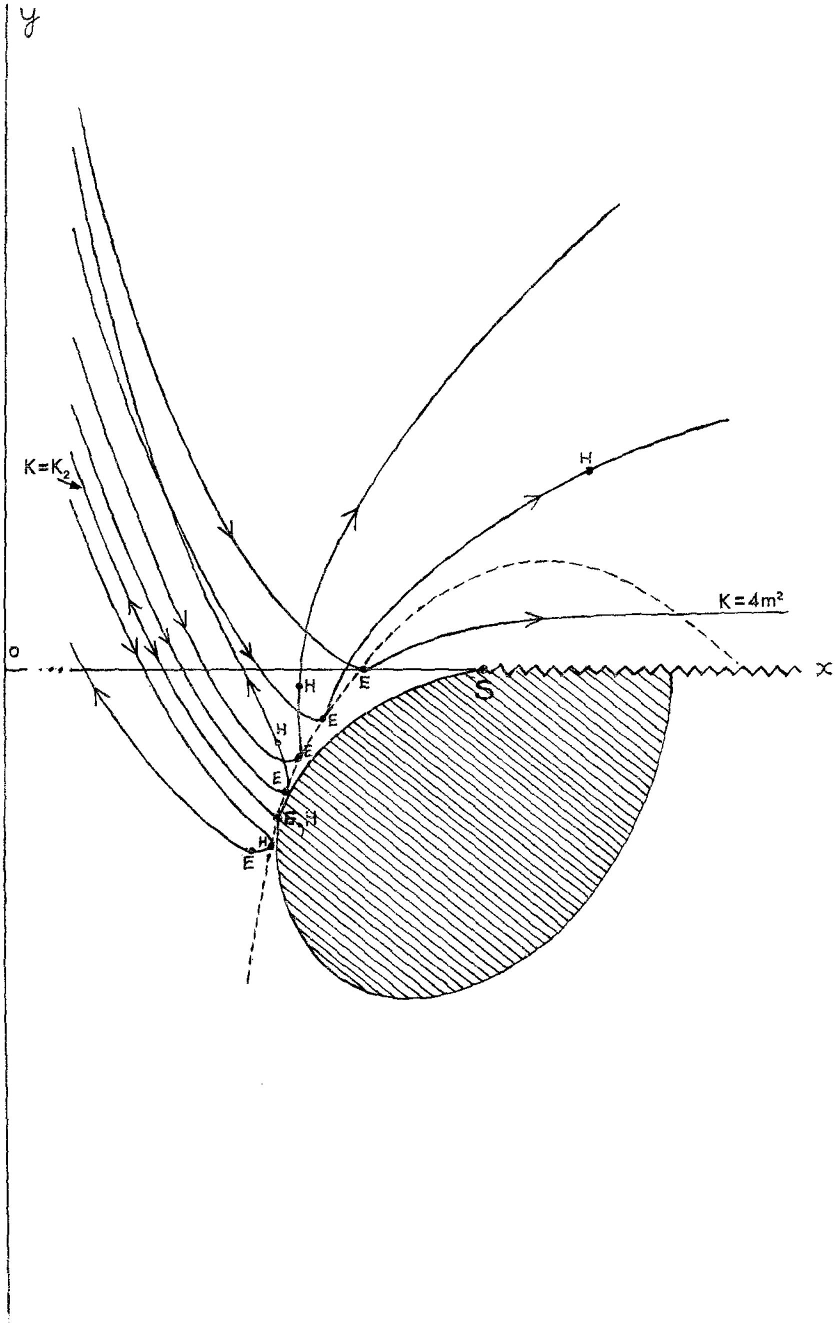

Figure 7—The solutions around $K = K_2$. $\partial/\partial\phi$ is time-like in the shaded region.

along the outward path in the (r, θ, ϕ) space as the sign of $\dot{\phi}$ remains unchanged. None the less our observer sees his own reflection!

For higher values of K the solutions meet E before H and reflect towards S. H flees down the curve away from E and when $K = 4m^2$ it has gone off to infinity. This solution bounces off the disc $r = 0$ and is asymptotically parallel to an hyperbola given by a value of θ just less than $\pi/2$.

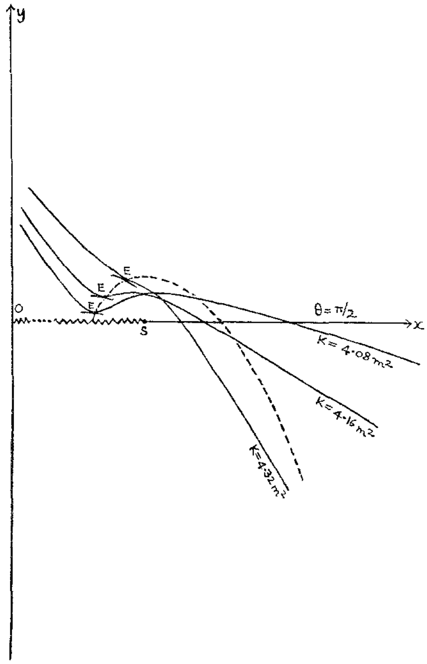

Figure 8—$K > 4m^2$: as K increases to $4{\cdot}32m^2$ solutions cross the equatorial plane progressively closer to S.

For K just above $4m^2$ the solutions begin to cross $\theta = \pi/2$, the point where this occurs moving rapidly towards S (Fig. 8) until $K \doteqdot 4{\cdot}32m^2$. Then the point moves away again, the curves enveloping a region which must remain invisible to the observer (Fig. 9). Now

$$\left\langle \frac{\partial}{\partial\phi}, \frac{\partial}{\partial\phi} \right\rangle = -\sin^2\theta \frac{[(r^2 + a^2)^2 - \Delta \sin^2\theta]}{r^2 + a^2\cos^2\theta} \tag{14}$$

so $\partial/\partial\phi$ is time-like when

$$\frac{(r^2 + a^2)^2}{\Delta} < a^2 \sin^2\theta$$

The circles $u = \text{constant}$, $r = \text{constant}$, $\theta = \text{constant}$ are closed time-like curves in this region (shaded in Fig. 7—with a symmetrical half beyond the cut) and closed time-like curves exist connecting any point in this region with any other point in the manifold (Carter, 1967, 1968). However, on the null geodesics, $R \geqslant 0$ and $\Theta \geqslant 0$ so $(r^2+a^2)^2/\Delta \geqslant K$ and $a^2 \sin^2\theta \leqslant K$. Therefore $(r^2+a^2)^2/\Delta \geqslant a^2\sin^2\theta$, and the null geodesics pass only through regions where $\partial/\partial\phi$ is space-like save for the one given by $K = K_2$ for which $(r^2+a^2)^2/\Delta$ and $a^2\sin^2\theta$ equal K_2 simultaneously at one point and hence $\partial/\partial\phi$ is null. Our observer sees himself reflected off the boundary of the

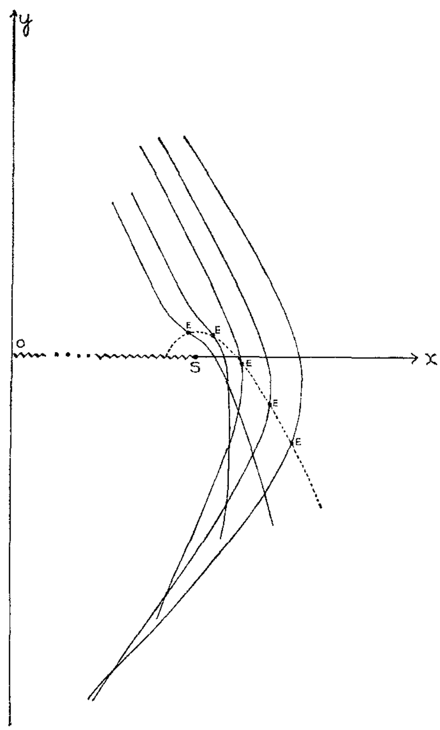

Figure 9—$K > 4{\cdot}32m^2$: the solutions envelope a region behind the ring.

region where $\partial/\partial\phi$ is time-like. If $\underline{p}$ is the 4-momentum of any physical particle in the region where $\partial/\partial\phi$ is time-like then $\langle \underline{p}, \partial/\partial\phi\rangle > 0$. But for a particle in geodesic motion $\langle \underline{p}, \partial/\partial\phi\rangle = -\Phi$, so the only geodesics which can enter this region are those with negative angular momentum.

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